

Indecomposables with smaller cohomological length in the derived category of gentle algebras

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Abstract

Bongartz and Ringel proved that there is no gaps in the sequence of lengths of indecomposable modules for the finite-dimensional algebras over algebraically closed fields. The present paper mainly study this “no gaps” theorem for the bounded derived module category $D^b(A)$ of a gentle algebra A : if there is an indecomposable object in $D^b(A)$ of cohomological length $l > 1$, then there exists an indecomposable with cohomological length $l - 1$.

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1 Introduction

Throughout this paper, k is an algebraically closed field, all algebras are connected basic finite-dimensional associative k -algebras with identity, and all modules are finite-dimensional right modules, unless stated otherwise. During the study of the representation theory of finite-dimensional algebras, the classification and distribution of indecomposable modules play an significant role. Besides the famous Brauer-Thrall conjectures [1, 5, 6, 8, 9], Bongartz and Ringel proved the following elegant theorem in [3, 7]:

Theorem 0. *Let A be an finite-dimensional algebra. If there is an indecomposable A -module of length $n > 1$, then there exists an indecomposable A -module of length $n - 1$.*

Since Happel [4], the bounded derived categories of finite-dimensional algebras have been studied widely. The classification and distribution of indecomposable objects in the bounded derived category is still an important theme in representation theory of algebras. Vossieck defined the *derived discrete algebras* using the cohomology dimension vector of objects in the bounded derived category and classified the derived discrete algebras: piecewise hereditary algebras of Dynkin type and a special class of gentle algebras, in his paper [10]. Due to [11], some numerical invariants, i.e. the cohomological length, width, and range of a complex in bounded derived category are introduced: let A be a finite-dimensional algebra with $D^b(A)$ the bounded derived module category, the *cohomological length*, *cohomological width*, *cohomological range* of a complex $X^\bullet \in D^b(A)$ are

$$\text{hl}(X^\bullet) := \max\{\dim H^i(X^\bullet) \mid i \in \mathbb{Z}\},$$

$$\text{hw}(X^\bullet) := \max\{j - i + 1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\},$$

$$\text{hr}(X^\bullet) := \text{hl}(X^\bullet) \cdot \text{hw}(X^\bullet),$$

respectively. Moreover, the Brauer-Thrall type theorems for derived module categories are established. As pointed out in [11, Question], it is natural to consider the derived version of Bongartz-Ringel's theorem and ask whether there are no gaps in the sequence of cohomological lengths (ranges) of indecomposable objects in $D^b(A)$.

Question I *Is there an indecomposable object in $D^b(A)$ of cohomological length $l - 1$ if there is one of cohomological length $l \geq 2$?*

Question II *Is there an indecomposable object in $D^b(A)$ of cohomological range $r - 1$ if there is one of cohomological range $r \geq 2$?*

Note that there is a full embedding from the category of finite-dimensional right A -modules $\text{mod} A$ into $D^b(A)$ which sends a module to the corresponding stalk complex. Obviously, the dimension of an A -module M is equal to the cohomological length and the cohomological range of the stalk complex M . So the questions are evidently true for representation-infinite algebras. However, it seems difficult to give answers for general finite-dimensional algebras to above questions since we know little about the description of indecomposables in the bounded derived category.

In this paper, we provide a partial answer to the above questions: the question I holds for the gentle algebras but the question II does not. To be precise, we prove that there is no gaps in the sequence of cohomological

lengths of indecomposables in the bounded derived category of gentle algebras. In addition, we construct a gentle algebra A_0 such that there is an indecomposable object in $D^b(A_0)$ of cohomological range r_0 but no indecomposable object with cohomological range $r_0 - 1$. Our result relies heavily on the constructions of indecomposables in the bounded derived category of gentle algebras due to Bekkert and Merklen [2].

The paper is organized as follows: in Section 2, we shall recall the constructions of indecomposable objects in bounded derived category of gentle algebras. In Section 3, we shall prove the main theorem of this paper. Finally, we provide a negative answer by a gentle algebra to Question II in the last section.

2 Indecomposables in bounded derived category of gentle algebras

In this section, we mainly recall the description of the indecomposable objects in the bounded derived category of gentle algebras from [2].

Let A be an algebra of form kQ/I , where $Q = (Q_0, Q_1, s, t)$ be a finite quiver with Q_0, Q_1 the set of vertices and arrows respectively as usual, and I be an admissible ideal of kQ . Recall that $A = kQ/I$ is a gentle algebra if

- (1) the number of arrows with a given source (resp. target) is at most two;
- (2) for any arrow $\alpha \in Q_1$, there is at most one arrow $\beta \in Q_1$ such that $s(\alpha) = t(\beta)$ (resp. $t(\alpha) = s(\beta)$) and $\beta\alpha \in I$ (resp. $\alpha\beta \in I$).
- (3) for any arrow $\alpha \in Q_1$, there is at most one arrow $\gamma \in Q_1$ such that $s(\alpha) = t(\gamma)$ (resp. $t(\alpha) = s(\gamma)$) and $\gamma\alpha \notin I$ (resp. $\alpha\gamma \notin I$).
- (4) I is generated by a set of path of length two.

Let $A = kQ/I$ be a gentle algebra. We need to recall some notations. For a path $p = \alpha_1\alpha_2 \cdots \alpha_r$ with $\alpha_i \in Q_1$, we say its length $l(p) = r$. Let $\mathbf{Pa}_{\geq 1}$ be the set of all path in kQ/I of length greater than 1. For any arrow $\alpha \in Q_1$, we denote by α^{-1} its formal inverse with $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. For a path $p = \alpha_1\alpha_2 \cdots \alpha_r$, its inverse $p^{-1} = \alpha_r^{-1}\alpha_{r-1}^{-1} \cdots \alpha_1^{-1}$. A sequence $w = w_1w_2 \cdots w_n$ is a *walk* (resp. a *generalized walk*) if each w_i is of form p or p^{-1} with $p \in Q_1$ (resp. $p \in \mathbf{Pa}_{\geq 1}$), and $s(w_{i+1}) = t(w_i)$ for $i = 1, 2, \dots, n-1$.

We denote by \mathbf{St} the set of all walks $w = w_1w_2 \cdots w_n$ such that $w_{i+1} \neq w_i^{-1}$ for each $1 \leq i < n$ and no subword of w or w^{-1} lies in I . We call the

an element in **St** a *string*. By $\overline{\mathbf{Gst}}$ we denote the set of all generalized walks such that

- (1) $w_i w_{i+1} \in I$ if $w_i, w_{i+1} \in \mathbf{Pa}_{\geq 1}$;
- (2) $w_{i+1}^{-1} w_i^{-1} \in I$ if $w_i^{-1}, w_{i+1}^{-1} \in \mathbf{Pa}_{\geq 1}$;
- (3) $w_i w_{i+1} \in \mathbf{St}$ otherwise.

We write \mathbf{Gst} the set consisting of all trivial paths and the representatives of $\overline{\mathbf{Gst}}$ modulo the relation $w \sim w^{-1}$. An element $w = w_1 w_2 \cdots w_n$ in \mathbf{Gst} is called a *generalized string* of width n .

Generalized bands are special generalized strings. Before its definition, we need the following notation. Let $w = w_1 w_2 \cdots w_n$ be a generalized string, set $\mu_w(0) = 0$, $\mu_w(i) = \mu_w(i-1) - 1$ if $w_i \in \mathbf{Pa}_{\geq 1}$ and $\mu_w(i) = \mu_w(i-1) + 1$ otherwise. Suppose \overline{GBa} is the set of all generalized walk $w = w_1 w_2 \cdots w_n$ such that

- (1) $s(w_1) = t(w_n)$;
- (2) $\mu_w(n) = \mu_w(0) = 0$;
- (3) $w^2 = w_1 w_2 \cdots w_n w_1 w_2 \cdots w_n \in \overline{Gst}$.

We denote by \mathbf{Gba} the set consisting of the representatives of \overline{GBa} modulo the relation identifies a generalized walk with its rotations and inverse. We call an element in \mathbf{Gba} a *generalized band*.

By the description of Bekkert and Merklen [2], a generalized string in $A = kQ/I$ corresponds to a unique indecomposable object of bounded homotopy category $K^b(\text{proj} A)$ up to shift, while a generalized band w corresponds to a family of indecomposables $\{P_{w,f}^\bullet \mid f = (x - \lambda)^d \in k[x], \lambda \in k^*, d > 0\}$ in $K^b(\text{proj} A)$ with same cohomologies. Thus A is derived discrete if and only if A contains no generalized bands, see [2, 10].

Let α be a path in $\mathbf{Pa}_{\geq 1}$. Then it induces a morphism $P(\alpha)$ from $P_{t(\alpha)}$ to $P_{s(\alpha)}$ by left multiplication, where $P_i = e_i A$. Precisely, $P(\alpha)(u) = \alpha u$ for any $u \in kQ/I$.

Definition 1. Let $w = w_1 w_2 \cdots w_n$ be a generalized string. Then the projective complex $P_w^\bullet = \cdots \xrightarrow{d_w^{i-1}} P_w^i \xrightarrow{d_w^i} P_w^{i+1} \xrightarrow{d_w^{i+1}} \cdots$ is defined as follows. The module on the i -th component

$$P_w^i = \bigoplus_{j=0}^n \delta(\mu_w(j), i) P_{c(j)},$$

where δ is the Kronecker sign, $c(j) = s(w_{j+1})$ for $j < n$ and $c(n) = t(w_n)$.

The differential $d_w^i = (d_{j,k}^i)$ and

$$d_{j,k}^i = \begin{cases} P(w_j), & \text{if } w_j \in \mathbf{Pa}_{\geq 1}, \mu_w(j) = i, k = j - 1; \\ P(w_{j+1}^{-1}), & \text{if } w_{j+1}^{-1} \in \mathbf{Pa}_{\geq 1}, \mu_w(j) = i, k = j + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2. Let $w = w_1 w_2 \cdots w_n$ be a generalized band. Then for any $f = (x - \lambda)^d \in k[x]$, $\lambda \in k^*$, $d > 0$, the projective complex

$$P_{w,f}^\bullet = \cdots \xrightarrow{d_w^{i-1}} P_{w,f}^i \xrightarrow{d_w^i} P_{w,f}^{i+1} \xrightarrow{d_w^{i+1}} \cdots$$

is defined as follows. The module on the i -th component

$$P_{w,f}^i = \bigoplus_{j=0}^{n-1} \delta(\mu_w(j), i) P_{c(j)}^d.$$

The differential $d_w^i = (d_{j,k}^i)$ and

$$d_{j,k}^i = \begin{cases} P(w_j) \mathbf{Id}_d, & \text{if } w_j \in \mathbf{Pa}_{\geq 1}, \mu_w(j) = i, k = j - 1; \\ P(w_{j+1}^{-1}) \mathbf{Id}_d, & \text{if } w_{j+1}^{-1} \in \mathbf{Pa}_{\geq 1}, \mu_w(j) = i, k = j + 1; \\ P(w_n) J_{\lambda,d}, & \text{if } w_n \in \mathbf{Pa}_{\geq 1}, \mu_w(n) = 0 = i, k = n - 1; \\ P(w_n^{-1}) J_{\lambda,d}, & \text{if } w_n^{-1} \in \mathbf{Pa}_{\geq 1}, \mu_w(n - 1) = i, k = 0; \\ 0, & \text{otherwise,} \end{cases}$$

where $J_{\lambda,d}$ the upper triangular $d \times d$ Jordan block with eigenvalue $\lambda \in k^*$.

Note that the definitions above are slightly different from ones in [2] since we consider right projective modules throughout this paper.

The following theorem from [2, Theorem 3] provides an explicit description of the indecomposables in bounded derived category $D^b(A)$.

Theorem 1. Let $A = kQ/I$ be a gentle algebra with $[-1]$ the shift functor in $D^b(A)$. Then the set of indecomposable objects in $K^b(\text{proj} A)$ is

$$\{P_w^\bullet[i] \mid w \in \mathbf{Gst}, i \in \mathbb{Z}\} \cup \{P_{w,f}^\bullet[i] \mid w \in \mathbf{Gba}, f = (x - \lambda)^d, \lambda \in k^*, d > 0, i \in \mathbb{Z}\}.$$

Moreover, the indecomposables in $K^{-,b}(\text{proj} A) \setminus K^b(\text{proj} A)$ is of form $\beta(P_w^\bullet)$ for $w \in \mathbf{Gst}$ with certain conditions.

3 The question I for gentle algebras

In this section, we will discuss the cohomological lengths of the indecomposables in the bounded derived category of gentle algebras. Indeed, we prove the following theorem.

Theorem 2. *Let A be a gentle algebra. If there is an indecomposable object in $D^b(A)$ of cohomological length $l > 1$, then there exists an indecomposable with cohomological length $l - 1$.*

Before the proof, we need some preparations. First, we recall the definitions of some numerical invariants for finite-dimensional algebras introduced in [11].

Definition 3. *Let A be a finite-dimensional algebra with $D^b(A)$ the bounded derived category. The cohomological length of a complex $X^\bullet \in D^b(A)$ is*

$$\text{hl}(X^\bullet) := \max\{\dim H^i(X^\bullet) \mid i \in \mathbb{Z}\}.$$

As well known, there is a full embedding of $\text{mod } A$ into $D^b(A)$ which sends a module to the corresponding stalk complex and the cohomological length of the stalk complex M equals to dimension of an A -module M . If A is representation-infinite, due to the truth of Brauer-Thrall conjecture I [1, 8], i.e., there exist indecomposable A -modules of arbitrary large dimensions, then the *global cohomological length* of A

$$\text{gl.h}A := \sup\{\text{hl}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\} = \infty.$$

Moreover, by the Bongartz and Ringel's theorem, Theorem 2 also holds for representation-infinite algebras.

Definition 4. The *cohomological width* of a complex $X^\bullet \in D^b(A)$ is

$$\text{hw}(X^\bullet) := \max\{j - i + 1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\},$$

and the *cohomological range* of X^\bullet is

$$\text{hr}(X^\bullet) := \text{hl}(X^\bullet) \cdot \text{hw}(X^\bullet).$$

Since the cohomological width of a stalk complex is one, the cohomological range of a stalk complex is precisely the cohomological length. Thus, there is also no gaps in the sequence of cohomological ranges of indecomposable objects in $D^b(A)$ if A is representation-infinite. Moreover, the cohomological length, width and range are invariant under shifts and isomorphisms.

The following lemma due to [11, Proposition 2] sets up the connection between the indecomposable objects in $K^b(\text{proj } A)$ and those in $K^{-,b}(\text{proj } A)$.

Lemma 1. *Let $P^\bullet \in K^{-,b}(\text{proj} A)$ be a minimal complex and $-n := \min\{i \in \mathbb{Z} \mid H^i(P^\bullet) \neq 0\}$. Then P^\bullet is indecomposable if and only if so is the brutal truncation $\sigma_{\geq j}(P^\bullet) \in K^b(\text{proj} A)$ for $j < -n$.*

Let A be a finite-dimensional algebra and $P^\bullet \in K^b(\text{proj} A)$ an indecomposable minimal complex of form

$$P^\bullet = 0 \longrightarrow P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \xrightarrow{d^{-n+1}} \dots \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \longrightarrow 0.$$

Now we can construct an minimal object in $D^b(A)$ by eliminating the cohomology of minimal degree. Suppose $H^{-n}(P^\bullet) \cong \text{Ker} d^{-n}$, we take a minimal projective resolution of $\text{Ker} d^{-n}$, say

$$P'^\bullet = \dots \longrightarrow P^{-n-2} \xrightarrow{d^{-n-2}} P^{-n-1} \longrightarrow 0.$$

Gluing P'^\bullet and P^\bullet together, we get a minimal complex

$$\beta(P^\bullet) = \dots \longrightarrow P^{-n-2} \xrightarrow{d^{-n-2}} P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} \dots \xrightarrow{d^{-1}} P^0 \longrightarrow 0,$$

where d^{-n-1} is the composition $P^{-n-1} \twoheadrightarrow \text{Ker} d^{-n} \hookrightarrow P^{-n}$. Note that $H^{-n}(\beta(P^\bullet)) = 0$, and $H^j(\beta(P^\bullet)) = H^j(P^\bullet)$ for $j \neq -n$.

Lemma 2. *Keep the notations as above. Then $\beta(P^\bullet)$ is indecomposable.*

Proof. If $H^{-n}(P^\bullet) = 0$, then $\beta(P^\bullet) = P^\bullet$ and the statement follows. Now suppose $H^{-n}(P^\bullet) \neq 0$. Since P^\bullet is the brutal truncation $\sigma_{\geq -n}(\beta(P^\bullet))$, which is indecomposable and $H^i(\beta(P^\bullet)) = 0$ for all $i \leq -n$, $\beta(P^\bullet)$ is indecomposable by Lemma 1. \square

Let A be a gentle algebra. By Theorem 1, any $P^\bullet \in D^b(A)$ is of form P_w^\bullet determined by a generalized string w , or of form $\beta(P_w^\bullet)$ for some generalized string w , or of form $P^\bullet = P_{w,f}^\bullet$ determined by a generalized band w . Thus we divide the proof of Theorem 2 into three theorems as follows and their proofs depend strongly on the description of the indecomposables in the bounded derived category of gentle algebras due to Bekkert and Merklen [2].

Theorem 3. *Let A be a gentle algebra. If there is an indecomposable $P_w^\bullet \in K^b(\text{proj} A)$ determined by a generalized string w such that $\text{hl}(P^\bullet) = l > 1$, then there is an indecomposable $P'^\bullet \in D^b(A)$ with $\text{hl}(P'^\bullet) = l - 1$.*

Proof. We shall divide the proof into three cases.

Case 1: Let $w = w_1 w_2 \cdots w_n$ be a one-sided generalized string, i.e. $w_i \in \mathbf{Pa}_{\geq 1}$ for all $1 \leq i \leq n$, or $w_i^{-1} \in \mathbf{Pa}_{\geq 1}$ for all $1 \leq i \leq n$. Without loss of generality, we assume $w_i \in \mathbf{Pa}_{\geq 1}$ for all $1 \leq i \leq n$ (Otherwise, we can consider the generalized string w^{-1} , and they determine the same complex). Let P^\bullet be the complex determined by w of form

$$P_w^\bullet = 0 \longrightarrow P_{t(w_n)} \xrightarrow{P(w_n)} P_{t(w_{n-1})} \xrightarrow{P(w_{n-1})} \cdots \xrightarrow{P(w_2)} P_{t(w_1)} \xrightarrow{P(w_1)} P_{s(w_1)} \longrightarrow 0,$$

where $P_{s(w_1)}$ lies in the 0-th component. Thus,

$$\dim H^0(P_w^\bullet) = \dim P_{s(w_1)} - \dim \operatorname{Im} P(w_1) = \dim P_{s(w_1)} - \dim w_1 P_{t(w_1)}.$$

Note that for any $p \in \mathbf{Pa}_{\geq 1}$, there is a unique maximal path $\tilde{p} = p\hat{p}$ starting with p . Moreover, by the definition of gentle algebras, there may be two maximal path from a fixed point. Thus, besides the path \tilde{p} , there may be another maximal path \check{p} begin with the start point $s(p)$ of p . If this is not the case, we write $l(\check{p}) = 0$. Then we have

$$\dim H^0(P_w^\bullet) = (l(\widetilde{w_1}) + l(\check{w_1}) + 1) - (l(\widehat{w_1}) + 1) = l(w_1) + l(\check{w_1}).$$

For any $1 \leq i \leq n-1$,

$$\begin{aligned} \dim H^{-i}(P_w^\bullet) &= \dim \operatorname{Ker} P(w_i) - \dim \operatorname{Im} P(w_{i+1}) \\ &= l(\widetilde{w_{i+1}}) - (l(\widehat{w_{i+1}}) + 1) \\ &= l(w_{i+1}) - 1. \end{aligned}$$

Similarly,

$$\begin{aligned} \dim H^{-n}(P_w^\bullet) &= \dim \operatorname{Ker} P(w_n) = \#\{p \in \mathbf{Pa}_{\geq 1} \mid w_n p = 0\} \\ &= \begin{cases} 0, & \text{if no arrows } \alpha \text{ such that } w_n \alpha = 0; \\ l(\tilde{\alpha}), & \text{if there is an arrow } \alpha \text{ such that } w_n \alpha = 0. \end{cases} \end{aligned}$$

For any walk $p = p_1 p_2 \cdots p_l$ and any $j < p$, we write $\kappa_j^+(p) = p_{j+1} p_{j+2} \cdots p_l$ the walk cutting the first j arrows from path p along the positive direction. Similarly, we write $\kappa_j^-(p) = p_1 p_2 \cdots p_{l-j}$ the walk cutting the last j arrows from path p along the negative direction. Now we suppose

$$i = \max\{j \mid \dim H^{-j}(P_w^\bullet) = \operatorname{hl}(P_w^\bullet); 0 \leq j \leq n\}.$$

Note that $\dim H^{-j}(P_w^\bullet) < \dim H^{-i}(P_w^\bullet)$ for any $j > i$ by definition. For $0 \leq i \leq n$, we shall construct an indecomposable P'^\bullet , such that $\operatorname{hl}(P'^\bullet) = \operatorname{hl}(P_w^\bullet) - 1$.

(1) If $i = 0$, then $\dim H^j(P_w^\bullet) < \dim H^0(P_w^\bullet)$ for any $j \neq 0$. Now we want to obtain a generalized string which determines a projective complex with the cohomological length equals to $\dim H^0(P_w^\bullet) - 1 = l(w_1) + l(\tilde{w}_1) - 1$.

If $l(\tilde{w}_1) = 0$, namely, \tilde{w}_1 is the unique maximal path starting from $s(w_1)$, then we get a generalized string $w' = \kappa_1^+(w_1)w_2 \cdots w_n$ by the cutting from positive direction. Now if there is a unique maximal path beginning with $s(w') = s(\kappa_1^+(w_1))$, then

$$\dim H^0(P_{w'}^\bullet) = l(\kappa_1^+(w_1)) = l(w_1) - 1 = \dim H^0(P_w^\bullet) - 1,$$

and the cohomologies of other degrees remain unchanged. Thus $P'^\bullet = P_{w'}^\bullet$ is as required with $\text{hl}(P'^\bullet) = l - 1$. If there is another arrow p starting from $s(w')$ besides w' , then we set $w'' = \overline{p^{-1}}\kappa_1^+(w_1)w_2 \cdots w_n$, where for a path p , $\overline{p^{-1}}$ denote the generalized string $\cdots p_3^{-1}p_2^{-1}p_1^{-1}p^{-1}$ such that $p_i \in Q_1$ with the maximal width. Note that $pp_1 \in I$, $p_i p_{i+1} \in I$ for $i \geq 1$, and thus $\overline{p^{-1}} = p^{-1}$ if there is no such arrow p_1 that $pp_1 \in I$. Indeed, the complex $P_{w''}^\bullet$ determined by w'' can be illustrated as follows

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_{t(w_n)} & \xrightarrow{P(w_n)} & P_{t(w_{n-1})} & \xrightarrow{P(w_{n-1})} & \cdots \xrightarrow{P(w_2)} P_{t(w_1)} \xrightarrow{P(\kappa_1^+(w_1))} P_{s(\kappa_1^+(w_1))} \longrightarrow 0 \\ & & & & & & \nearrow P(p) \\ & & & & \cdots \xrightarrow{P(p_1)} & P_{t(p)} & \end{array}$$

with $P_{s(\kappa_1^+(w_1))}$ on the 0-th component. Now we calculate the dimension of cohomologies of $P_{w''}^\bullet$.

$$\begin{aligned} \dim H^0(P_{w''}^\bullet) &= \dim P_{s(\kappa_1^+(w_1))} - \dim \text{Im}(P(\kappa_1^+(w_1)), P(p)) \\ &= \widehat{l(\kappa_1^+(w_1))} + l(\tilde{p}) + 1 - (\widehat{l(\kappa_1^+(w_1))} + 1) - (l(\tilde{p}) + 1) \quad (*) \\ &= l(\kappa_1^+(w_1)) + l(p) - 1 = l(\kappa_1^+(w_1)) \\ &= l(w_1) - 1 = \dim H^0(P_w^\bullet) - 1. \end{aligned}$$

Moreover, the cohomologies of other degrees remain unchanged since $p_i \in Q_1$. Indeed, in the construction of w'' , the reason for gluing $\overline{p^{-1}}$ and w' together is to reduce the dimension of the 0-th cohomology by 1 without changing the dimension of cohomologies of other degrees. Thus $P'^\bullet = P_{w''}^\bullet$ is as required with $\text{hl}(P'^\bullet) = l - 1$.

If $l(\tilde{w}_1) = a > 0$, then we set $w' = \overline{\tilde{w}_1}^{-1}w_1w_2 \cdots w_n$. By the calculation as in the equations (*), $\dim H^0(P_{w'}^\bullet) = l(w_1) + l(\tilde{w}_1) - 1 = \dim H^0(P_w^\bullet) - 1$, and the cohomologies of other degrees remain unchanged. Thus $P'^\bullet = P_{w'}^\bullet$ is the complex as required.

(2) If $1 \leq i \leq n - 1$ and $l(w_{i+1}) > 2$, then we set the generalized string $w' = \kappa_2^+(w_{i+1})w_{i+2} \cdots w_n$ obtained by cutting from the positive direction.

Similar with the discussion in the case (1), if $\kappa_2^+(w_{i+1})$ is the unique maximal path beginning with $s(\kappa_2^+(w_{i+1}))$, then w' determines an indecomposable $P_{w'}^\bullet$ such that

$$\begin{aligned}\dim H^{-i}(P_{w'}^\bullet[-i]) &= \dim H^0(P_{w'}^\bullet) = l(\kappa_2^+(w_{i+1})) \\ &= l(w_{i+1}) - 2 = \dim H^{-i}(P_w^\bullet) - 1 = \text{hl}(P_w^\bullet) - 1,\end{aligned}$$

and $\dim H^{-j}(P_{w'}^\bullet[-i]) = 0$ for any $j < i$, $\dim H^{-j}(P_{w'}^\bullet[-i]) \leq \dim H^{-j}(P_w^\bullet) < \dim H^{-i}(P_w^\bullet)$ for any $j > i$. So $P_{w'}^\bullet[-i]$ is the complex as required in this case. If there is another arrow p beginning with $s(\kappa_2^+(w_{i+1}))$, then we set $w'' = \overline{p^{-1}}\kappa_2^+(w_{i+1})w_{i+2} \cdots w_n$. By a similar calculation as in Case (1), $P'^\bullet = P_{w''}^\bullet$ satisfies $\text{hl}(P'^\bullet) = \text{hl}(P_w^\bullet) - 1$.

(3) Finally, for the case $i = n$, if there is no arrows α such that $w_n\alpha = 0$, then $\text{hl}(P_w^\bullet) = 0$, which is impossible. Assume there is an arrow α with $w_n\alpha = 0$ and $l(\tilde{\alpha}) > 1$, then we choose the generalized string $w' = \kappa_1^+(\tilde{\alpha})$. With a similar discussion as above, if there is a unique path beginning with $s(w')$, then w' determines the indecomposable object $P_{w'}^\bullet$. Set the indecomposable object $P'^\bullet = \beta(P_{w'}^\bullet)$, then we have $\dim H^{-n}(P'^\bullet[-n]) = \dim H^0(P_{w'}^\bullet) = l(\tilde{\alpha}) - 1 = \dim H^{-n}(P_w^\bullet) - 1 = \text{hl}(P_w^\bullet)$, and the cohomologies of other degrees vanish. Therefore, $\text{hl}(P'^\bullet) = \text{hl}(P_w^\bullet) - 1$. While if there is another arrow p beginning with the starting point of w' , then set $w'' = p^{-1}w' = p^{-1}\kappa_1^+(\tilde{\alpha})$ and $P'^\bullet = \beta(P_{w''}^\bullet)$. Thus $\dim H^{-n}(P'^\bullet[-n]) = \dim H^0(P_{w''}^\bullet) = l(\kappa_1^+(\tilde{\alpha})) + l(p) - 1 = l(\tilde{\alpha}) - 1 = \dim H^{-n}(P_w^\bullet) - 1 = \text{hl}(P_w^\bullet)$, and the cohomologies of other degrees vanish.

In the above three cases, the indecomposable object P'^\bullet are constructed base on the generalized string obtained by cutting from the positive direction. Indeed, in each case, we can also obtain another indecomposable object as required by cutting the generalized strings from the negative direction. We shall take the case (2) above for example. First, we set

$$i = \min\{j \mid \dim H^{-j}(P_w^\bullet) = \text{hl}(P_w^\bullet); 0 \leq j \leq n\}.$$

Now, we need to reduce the dimension of i -th cohomology by 1 and eliminate the j -th cohomology for $j < -i$. We get a generalized string $w' = w_1 \cdots w_i \kappa_1^-(w_{i+1})$ by cutting from the negative direction. As in the case (1), we glue w' and a generalized string together if needed to eliminate the cohomology at certain degree. To be precise, if there is no arrow α such that $\kappa_1^-(w_{i+1})\alpha \in I$, then

$$\dim H^{-i}(P_{w'}^\bullet) = l(\kappa_1^-(w_{i+1})) - 1 = l(w_{i+1}) - 2 = \dim H^{-i}(P_w^\bullet) - 1.$$

Moreover, $\dim H^{-j}(P_{w'}^\bullet) = 0$ for $j > i$, and $\dim H^{-j}(P_{w'}^\bullet) < \dim H^{-i}(P_w^\bullet)$ for $j < i$ by assumption on i . That means, $P^\bullet = P_{w'}^\bullet$ is also an indecomposable object with $\text{hl}(P^\bullet) = \text{hl}(P^\bullet) - 1$ as required. If there is an arrow α with $\kappa_1^-(w_{i+1})\alpha \in I$, then we set $w'' = w_1 \cdots w_i \kappa_1^-(w_{i+1})\underline{\alpha}$, where $\underline{\alpha}$ denote the generalized string $\alpha\alpha_1\alpha_2 \cdots$ of maximal width with $\alpha_i \in \mathbf{Pa}_1$. Then by a similar calculation, $P^\bullet = P_{w''}^\bullet$ is also an indecomposable object with $\text{hl}(P^\bullet) = \text{hl}(P^\bullet) - 1$ as required. Note that in this case, $P^\bullet = P_{w''}^\bullet = \beta(P_{w'}^\bullet)$, where β is the functor introduced in Lemma 2.

Case 2: Let $w = w_1 w_2 \cdots w_n$ be a generalized string. Without loss of generality, assume that $w_1^{-1}, w_2^{-1}, \dots, w_q^{-1} \in \mathbf{Pa}_{\geq 1}$ and $w_{q+1}, w_{q+2}, \dots, w_r \in \mathbf{Pa}_{\geq 1}$, while $w_{r+1}^{-1} \in \mathbf{Pa}_{\geq 1}$. Then w determines the indecomposable object P_w^\bullet of form

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P_{s(w_1)} & \xrightarrow{P(w_1^{-1})} & \cdots & \longrightarrow & P_{s(w_k)} \xrightarrow{P(w_k^{-1})} P_{s(w_{k+1})} \xrightarrow{P(w_{k+1}^{-1})} \cdots \xrightarrow{P(w_{q-1}^{-1})} P_{s(w_q)} \xrightarrow{P(w_q^{-1})} P_{t(w_q)} \\
 & & & & & & & \nearrow P(w_{q+1}) \\
 & & P_{t(w_r)} & \xrightarrow{P(w_r)} P_{t(w_{r-1})} & \xrightarrow{P(w_{r-1})} \cdots \xrightarrow{P(w_{q+2})} & P_{t(w_{q+1})} & \\
 & & \searrow P(w_{r+1}^{-1}) & & & & \\
 & & & P_{s(w_{r+2})} & \xrightarrow{P(w_{r+2}^{-1})} & \cdots &
 \end{array}$$

where $P_{s(w_1)}$ lies in the 0-th component.

As illustrated above, there may be more than one indecomposable projective direct summands at a component. Note that at each component, we can order these indecomposable projective direct summands *where has nonzero cohomology* along the generalized string w . For example, in the above diagram, suppose the projective module $P_{s(w_{k+1})}$ lies in the i -th component, then we write $P_w^i = P_1^i \oplus P_2^i \oplus P_3^i \oplus \cdots$, where $P_1^i = P_{s(w_{k+1})}$, $P_2^i = P_{t(w_{r-1})}$, \cdots since the cohomologies are nontrivial at these direct summands. Then the cohomology of the degree i is the direct summand of cohomologies at these projective direct summands.

Now, as in Case 1, we want to construct an indecomposable object P'^{\bullet} such that $\mathrm{hl}(P'^{\bullet}) = \mathrm{hl}(P_w^{\bullet}) - 1$. In order to reduce the dimension of cohomologies of i -th degree by 1, it suffices to reduce the dimension of cohomologies at the first projective direct summand of i -th degree. Indeed, we need to find the unique projective direct summand Q satisfying

- 1) It is the first direct projective summand of its component under the ordering as above;
- 2) It lies in j -th component such that $\dim H^j(P^\bullet) = \text{hl}(P^\bullet)$;

3) It is the farthest one from the starting point along the generalized string among these satisfying 1) and 2).

To construct an indecomposable object P'^{\bullet} such that $\text{hl}(P'^{\bullet}) = \text{hl}(P_w^{\bullet}) - 1$, we only need to construct such P'^{\bullet} by reducing the dimension of cohomology at Q by 1. By the analysis in Case 1, it is enough to consider the cases Q is the *backward turning points* as $P_{t(w_q)}$ or *forward turning point* as $P_{t(w_r)}$, since otherwise, we can reduce the dimension of cohomologies at Q by 1 via cutting the generalized string from positive side and then gluing suitable generalized string of form \bar{p} if needed, or cutting the generalized string from negative side and gluing suitable generalized string of form \underline{a} if needed.

Let $Q = P_{t(w_q)}$ be a backward turning point. Then the dimension of cohomology here

$$\begin{aligned} \dim H^{t(w_q)}(P_w^{\bullet}) &= \dim P(t(w_q)) - \dim \text{Im}(P(w_q^{-1}), P(w_{q+1})) \\ &= l(\widehat{w_{q+1}}) + l(\widehat{w_q^{-1}}) + 1 - (l(\widehat{w_{q+1}}) + 1) - (l(\widehat{w_q^{-1}}) + 1) \\ &= l(w_{q+1}) + l(w_q^{-1}) - 1. \end{aligned}$$

Set $w' = \kappa_1^+(w_q)w_{q+1} \cdots w_n$. As in Case 1(1), if there is an arrow p such that $\kappa_1^+(w_q)p \in I$, then we write $w'' = \overline{p^{-1}}\kappa_1^+(w_q)w_{q+1} \cdots w_n$, and $w'' = w'$ otherwise. We have $\dim H^{t(w_q)}(P_{w''}^{\bullet}) = \dim H^{t(w_q)}(P_w^{\bullet}) - 1$ and then $\text{hl}(P_{w''}^{\bullet}) = \text{hl}(P_w^{\bullet}) - 1$.

Similarly let $Q = P_{t(w_r)}$ be a forward turning point. Then the dimension of cohomology here

$$\begin{aligned} \dim H^{t(w_r)}(P_w^{\bullet}) &= \dim \text{Ker}(P(w_r), P(w_{r+1}^{-1}))^T \\ &= \dim(\text{Ker} P(w_r) \cap \text{Ker} P(w_{r+1}^{-1})) \\ &= 0, \end{aligned}$$

it is impossible by the definition of Q . □

Now we consider the indecomposable objects in $K^{-,b}(\text{proj} A) \setminus K^b(\text{proj} A)$.

Theorem 4. *Let A be a gentle algebra. If there is an indecomposable $P^{\bullet} \in K^{-,b}(\text{proj} A) \setminus K^b(\text{proj} A)$ such that $\text{hl}(P^{\bullet}) = l > 1$, then there is an indecomposable $P'^{\bullet} \in D^b(A)$ with $\text{hl}(P'^{\bullet}) = l - 1$.*

Proof. Since $P^{\bullet} \in K^{-,b}(\text{proj} A) \setminus K^b(\text{proj} A)$ is indecomposable, by Theorem 1, the brutal truncation $\sigma_{\geq j}(P^{\bullet}) \in K^b(\text{proj} A)$ is indecomposable for some $j \ll 0$, and $\sigma_{\geq j}(P^{\bullet}) = P_w^{\bullet}$ for some generalized string w . Now we can consider the complex P_w^{\bullet} base on the previous theorem. If $\dim H^j(P_w^{\bullet}) \leq l$, then $\text{hl}(P_w^{\bullet}) = l$ and the previous theorem implies the statement. Suppose

$\dim H^j(P_w^\bullet) > l$. By a similar analysis in the proof of previous theorem, we can find the unique projective direct summand Q which satisfies that it is the first direct projective summand, it lies in m -th component such that $\dim H^m(P^\bullet) = l$ and it is the farthest one from the starting point along w . Then we can construct P'^\bullet by reducing the dimension of cohomology at Q by 1. Note that maybe $\dim H^j(P'^\bullet)$ has the maximal dimension among the cohomologies of all degrees. Thus we can apply β to eliminate the cohomology of j -th degree if needed, and $H^i(\beta(P'^\bullet)) = 0$ for any $i < j$. Thus $\text{hl}(\beta(P'^\bullet)) = l - 1$ as required. \square

To finish the proof Theorem 2, we only need to prove the last case, i.e. for the indecomposable objects determined by generalized bands.

Theorem 5. *Let A be a gentle algebra. If there is an indecomposable $P^\bullet \in K^b(\text{proj} A)$ determined by a generalized band w such that $\text{hl}(P^\bullet) = l > 1$, then there is an indecomposable $P'^\bullet \in D^b(A)$ with $\text{hl}(P'^\bullet) = l - 1$.*

Proof. Let $w = w_1 w_2 \cdots w_n$ be a generalized band. We assume $w_1^{-1}, w_n \in \mathbf{Pa}_{\geq 1}$ and

$$\mu(0) = \mu(n) = \min\{\mu(i) \mid 0 \leq i \leq n\}$$

without loss of generality. Then w determines a family of indecomposable objects $\{P_{w,f}^\bullet \mid w \in \mathbf{Gba}, f = (x - \lambda)^d, \lambda \in k^*, d > 0, i \in \mathbb{Z}\}$, where $P_{w,f}^\bullet$ has the form of

$$\begin{array}{ccccccc} P_{s(w_1)}^d & \xrightarrow{P(w_1^{-1})\mathbf{I}_d} & P_{s(w_2)}^d & \longrightarrow & \cdots & \longrightarrow & P_{s(w_r)}^d \xrightarrow{P(w_r)\mathbf{I}_d} P_{t(w_r)}^d \\ & \searrow P(w_n)\mathbf{J}_{\lambda,d} & & & & & \nearrow P(w_{r+1})\mathbf{I}_d \\ & & P_{s(w_n)}^d & \longrightarrow & \cdots & \longrightarrow & P_{t(w_{r+1})}^d \end{array}$$

where $P_{s(w_1)}$ lies in the 0-th component.

By the previous two theorems, it is sufficient to find a generalized string w' such that $\text{hl}(\beta(P_{w'}^\bullet)) = \text{hl}(P_{w,f}^\bullet)$. We claim the generalized string $w' = (w_1 w_2 \cdots w_n)^d$ is the one as required. Roughly speaking, the complex $P_{w'}^\bullet$ can be seen as the one untying the band $P_{w,f}^\bullet$ into a string. Let P_w^\bullet be the indecomposable object determined by $w = w_1 w_2 \cdots w_n$ seen as a generalized string. Then for any $i \in \mathbb{Z}$ except $i = 0$,

$$\dim H^i(P_{w,f}^\bullet) = d \cdot \dim H^i(P_w^\bullet) = \dim H^i(\beta(P_{w'}^\bullet)).$$

Moreover, if $i = 0$, then

$$\dim H^0(P_{w,f}^\bullet) = \dim(\text{Ker } P(w_1^{-1})\mathbf{I}_d \cap \text{Ker } P(w_n)\mathbf{J}_{\lambda,d}) = 0 = \dim H^0(\beta(P_{w'}^\bullet))$$

Therefore, $\text{hl}(\beta(P_{w'}^\bullet)) = \text{hl}(P_{w,f}^\bullet)$ as claimed. \square

4 An negative answer to question II

In this section, we will construct a gentle algebra which provides a negative answer to Question II.

Let $A_0 = kQ/I$ be the gentle algebra defined by the quiver

$$\begin{array}{ccccccccc} & & 1 & & & & & & \\ & & \downarrow \alpha_1 & & & & & & \\ 3 & \xleftarrow{\alpha_2} & 2 & \xrightarrow{\alpha_3} & 4 & \xrightarrow{\alpha_4} & 5 & \xrightarrow{\alpha_5} & 6 & \xrightarrow{\alpha_6} & 7 \end{array}$$

and the admissible ideal generated by $\alpha_1\alpha_3$. Now we consider the indecomposable object P_w^\bullet determined by generalized string $w = \alpha_1$, where

$$P_w^\bullet = 0 \longrightarrow P_2 \xrightarrow{P(w)} P_1 \longrightarrow 0,$$

with P_1 in the 0-th component. Clearly, $\dim H^{-1}(P_w^\bullet) = 4$ and $\dim H^0(P_w^\bullet) = 1$. So $\text{hr}(P_w^\bullet) = \text{hl}(P_w^\bullet) \cdot \text{hw}(P_w^\bullet) = 8$.

Next we claim that there is no indecomposable object in $D^b(A_0)$ with cohomological range 7. Assume to the contrary that there is an indecomposable $P^\bullet \in K^b(\text{proj} A_0)$ with $\text{hr}(P^\bullet) = 7$, then $\text{hw}(P^\bullet) = 7$ or $\text{hl}(P^\bullet) = 7$. We shall show they are impossible. Indeed, by the description due to [2], the indecomposables in the $D^b(A_0)$ are determined by the generalized strings in A_0 . Since the indecomposables in $D^b(A_0)$ determined by the generalized strings, we have

$$\text{gl.hw} A_0 := \sup\{\text{hw}(X^\bullet) \mid X^\bullet \in D^b(A_0) \text{ is indecomposable}\} = 3.$$

Moreover, since any generalized string in A_0 is one-sided, each component of the indecomposable object $P_w^\bullet \in K^b(\text{proj} A_0)$ is indecomposable, and then

$$\text{gl.hl} A_0 := \sup\{\text{hl}(X^\bullet) \mid X^\bullet \in D^b(A_0) \text{ is indecomposable}\} \leq \dim P_2 = 6.$$

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